

Maximum order of triangle-free graphs with a given rank

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Abstract

The rank of a graph is defined to be the rank of its adjacency matrix. A graph is called reduced if it has no isolated vertices and no two vertices with the same set of neighbors. We determine the maximum order of reduced triangle-free graphs with a given rank and characterize all such graphs achieving the maximum order.

Keywords: rank, triangle-free graph, adjacency matrix

AMS Mathematics Subject Classification (2010): 05C50, 05C75, 15A03

1 Introduction

For a graph G , we denote by $V(G)$ the vertex set of G . The order of G is defined as $|V(G)|$. Let $V(G) = \{v_1, \dots, v_n\}$. The *adjacency matrix* of G is an $n \times n$ matrix $A(G)$ whose (i, j) -entry is 1 if v_i is adjacent to v_j and 0 otherwise. The *rank* of G , denoted by $\text{rank}(G)$, is the rank of $A(G)$.

For a vertex v of G , let $N(v)$ denote the set of all vertices of G adjacent to v . The *degree* of v is defined as $|N(v)|$. We say that G is *reduced* if it has no isolated vertex and no two vertices u, v with $N(u) = N(v)$. Indeed, adding an isolated vertex or introducing a new vertex with the same neighbor set as an existing vertex does not change the rank. Let $r \geq 2$ be an integer. It is straightforward to see that every reduced graph of rank r has at most $2^r - 1$ vertices [1]. Let $m(r)$ be the maximum

possible order of a reduced graph of rank r . Kotlov and Lovász [7] proved that there exists a constant c such that $m(r) \leq c \cdot 2^{r/2}$ and for any $r \geq 2$ they constructed a graph of rank r and order

$$\mu(r) = \begin{cases} 2^{(r+2)/2} - 2 & \text{if } r \text{ is even,} \\ 5 \cdot 2^{(r-3)/2} - 2 & \text{if } r > 1 \text{ is odd.} \end{cases}$$

Akbari, Cameron and Khosrovshahi [1] conjectured that in fact $m(r) = \mu(r)$. Haemers and Peeters [4] proved the conjecture for graphs containing an induced matching of size $r/2$ or an induced subgraph consisting of a matching of size $(r-3)/2$ and a cycle of length 3. Royle [8] proved that the rank of every reduced graph containing no path of length 3 as an induced subgraph is equal to the order.

We proved in [3] that every reduced tree of rank r has at most $t(r) = 3r/2 - 1$ vertices and characterized all reduced trees of rank r and order $t(r)$. It was also shown that every reduced bipartite graph of rank r has at most $b(r) = 2^{r/2} + r/2 - 1$ vertices and all reduced bipartite graphs achieving this bound were determined. Note that the rank of a bipartite graph is always even. In this article, we prove that every reduced non-bipartite triangle-free graph of rank r has at most $c(r) = 3 \cdot 2^{\lfloor r/2 \rfloor - 2} + \lfloor r/2 \rfloor$ vertices and characterize all reduced non-bipartite triangle-free graphs of rank r and order $c(r)$.

2 Preliminaries

For a graph G , a subset S of $V(G)$ with $|S| > 1$ is called a *duplication class* of G if $N(u) = N(v)$, for every $u, v \in S$. For a subset X of $V(G)$, the notation $G - X$ represents the subgraph obtained by removing the vertices in X from G .

Lemma 1. [6, 7] *For any reduced graph G , the following hold.*

- (i) *For every vertex $v \in V(G)$, $\text{rank}(G - N(v)) \leq \text{rank}(G) - 2$.*
- (ii) *For every non-adjacent vertices $u, v \in V(G)$, $\text{rank}(G - (N(u) \Delta N(v))) \leq \text{rank}(G) - 2$, where Δ denotes the symmetric difference.*

The following lemma has a key role in our proofs.

Lemma 2. *Let G be a reduced graph and H be an induced subgraph of G with the maximum possible order subject to $\text{rank}(H) < \text{rank}(G)$. Then $\text{rank}(H) \geq \text{rank}(G) - 2$ and the equality occurs if H is not reduced. Moreover, the following properties hold.*

- (i) $|V(G) \setminus V(H)| \leq \min\{|N(u) \Delta N(v)| \mid u, v \in V(G)\} \cup \{|N(u)| \mid u \in V(G)\}$.
- (ii) *If w is an isolated vertex of H , then $N(w) = V(G) \setminus V(H)$.*
- (iii) *Each duplication class of H has two elements and H has at most one isolated vertex.*

- (iv) One may label the duplication classes of H , if any, as $\{v_1, v'_1\}, \dots, \{v_s, v'_s\}$ so that there exist two disjoint sets T_1 and T_2 such that $V(G - H) = T_1 \cup T_2$, $T_1 \subseteq N(v_i) \setminus N(v'_i)$ and $T_2 \subseteq N(v'_i) \setminus N(v_i)$, for all $i \in \{1, \dots, s\}$.

Furthermore, if H is an induced subgraph of G with the maximum possible order subject to $\text{rank}(H) \leq \text{rank}(G) - 2$, then $\text{rank}(H) \geq \text{rank}(G) - 3$ and the properties (i)–(iv) also hold.

Proof. If H is an induced subgraph of G with the maximum possible order subject to $\text{rank}(H) < \text{rank}(G)$, then the statements (i)–(iv) can be found among the results of [6] and also [7]. In order to prove the rest of the assertion, we let H be an induced subgraph of G with the maximum possible order subject to $\text{rank}(H) \leq \text{rank}(G) - 2$. We first establish that $\text{rank}(H) \geq \text{rank}(G) - 3$. Assume that H_1 is an induced subgraph of G with the maximum possible order subject to $\text{rank}(H_1) < \text{rank}(G)$. If $\text{rank}(H_1) = \text{rank}(G) - 2$, then we clearly have $\text{rank}(H) = \text{rank}(H_1)$. Also, if $\text{rank}(H_1) = \text{rank}(G) - 1$, then by the first part of the lemma, H_1 is reduced and so $\text{rank}(H_2) \geq \text{rank}(G) - 3$, where H_2 is an induced subgraph of H_1 with the maximum possible order subject to $\text{rank}(H_2) < \text{rank}(H_1)$. It follows that $\text{rank}(H) \geq \text{rank}(G) - 3$. By the definition of H and by Lemma 1, (i) and hence (ii) is valid. For (iii), let H have a duplication class containing three distinct vertices x, y, z . Clearly, for every vertex $t \in V(G) \setminus V(H)$, at least one of the three symmetric differences of $N(x), N(y), N(z)$ does not contain t . This contradicts (i). The second statement of (iii) follows from (ii), since G is reduced. For (iv), note first that, by the definition of H , any vertex in $V(G) \setminus V(H)$ is adjacent to exactly one vertex in each duplication class, since for any duplication class $\{x, y\}$ in H , we have $N(x) \triangle N(y) \subseteq H$. If (iv) does not hold, then $A(G)$ contains

$$\left[\begin{array}{cc|cc|c|cc} & & & & & 1 & 1 \\ & & & & & 0 & 0 \\ & x & x & y & y & \star & \\ & & & & & 1 & 0 \\ & & & & & 0 & 1 \\ & & & & & \star & \star \\ \hline 1 & 0 & 1 & 0 & \star & 0 & \star \\ 1 & 0 & 0 & 1 & \star & \star & 0 \end{array} \right] \quad (1)$$

as a principle submatrix, where the upper-left corner of (1) is $A(H)$. This yields that $\text{rank}(H) \leq \text{rank}(G) - 4$, a contradiction. \square

For any graph G , a subset X of $V(G)$ is called *independent* if the induced subgraph on X has no edges. The maximum size of an independent set in a graph G is called the *independence number* of G and is denoted by $\alpha(G)$. We will make use of the following lemma which is an immediate consequence of the Plotkin bound [5, p. 58] from coding theory and was also established in [3] by a direct proof.

Lemma 3. *Let G be a graph of order n and let S be an independent set in G with $|S| \geq 2$. Then*

$$\min \{ |N(u) \triangle N(v)| \mid u, v \in S, u \neq v \} \leq \frac{|S|(n - |S|)}{2(|S| - 1)}.$$

In the following, we recall the Singleton bound [5, p. 71] from coding theory.

Theorem 4. *Let n be a positive integer and Ω be the set of all $(0, 1)$ -vectors of length n . Let C be a subset of Ω so that every pair of the vectors in C differ in at least d positions. Then $|C| \leq 2^{n-d+1}$. The equality occurs if and only if one of the following holds.*

- (i) $C = \Omega$.
- (ii) C is the set of all even weight vectors of Ω .
- (iii) C is the set of all odd weight vectors of Ω .
- (iv) C consists of two vectors which are different in all positions.

We will use \mathbf{j} for the all one vector.

Lemma 5. *Let C be a set of $(0, 1)$ -vectors of length $n \geq 5$ such that every two distinct vectors in C differ in at least 2 positions. Let M be the matrix whose columns are the vectors in C and suppose that \mathbf{j} is contained in the row space of M . Then $|C| \leq 5 \cdot 2^{n-4}$.*

Proof. Toward a contradiction, suppose that $|C| > 5 \cdot 2^{n-4}$. Let

$$(x_1, \dots, x_n)M = \mathbf{j}, \quad (2)$$

for some reals x_1, \dots, x_n . Let M' be the matrix constituted from the last $n-2$ rows of M and partition the columns of M' such that equal columns belong to the same part. Since the number of parts in the partition is at most 2^{n-2} and $5 \cdot 2^{n-4} > 2^{n-2}$, there is a part of size at least 2. Since every two distinct columns in M differ in at least 2 positions, we find two columns in M such that their entries are the same at all positions except for the first and the second positions. It follows from (2) that either $x_1 = x_2$ or $x_1 = -x_2$. By applying this argument to any pair of rows of M and a suitable ordering of the rows of M , we find that $x_1 = \dots = x_k = -x_{k+1} = \dots = -x_n$, for some k . Now, let N be the matrix obtained from M by subtracting \mathbf{j} from i th row of M , for all $i \in \{k+1, \dots, n\}$, and leaving the first k rows intact. We have $\mathbf{j}N = (n-k+1/x_1)\mathbf{j}$. This means that the column vectors of N have the same number of ones which in turn implies that $|C| \leq \binom{n}{\lfloor n/2 \rfloor}$. This contradicts $|C| > 5 \cdot 2^{n-4} \geq \binom{n}{\lfloor n/2 \rfloor}$, for $n \geq 5$. \square

It is an interesting problem to determine the best upper bound for $|C|$ in Lemma 5.

In [3], the maximum order of a reduced bipartite graph of rank r is determined. The graph attaining the maximum order is unique and is described as follows. Let B be a set of size n and \mathcal{B} be a family of subsets of B . The *incidence graph* (B, \mathcal{B}) is the bipartite graph with bipartition $\{B, \mathcal{B}\}$ so that the vertices $x \in B$ and $X \in \mathcal{B}$ are adjacent if and only if $x \in X$. If $\mathcal{P}(B)$ is the family of all nonempty subsets of B , then we denote the incidence graph $(B, \mathcal{P}(B))$ by \mathcal{B}_n . It is routine to verify that \mathcal{B}_n is a reduced bipartite graph of rank $2n$ and order $b(2n)$. Further, we denote by \mathcal{O}_n the incidence graph corresponding to the family of all subsets of B of odd size.

Theorem 6. [3] *The order of a reduced bipartite graph of rank r is at most $b(r) = 2^{r/2} + r/2 - 1$. Moreover, every reduced bipartite graph of rank r and order $b(r)$ is isomorphic to $\mathcal{B}_{r/2}$.*

3 Bipartite graphs

For a bipartite graph G with bipartition $\{X, Y\}$, the submatrix of $A(G)$ whose rows and columns are respectively indexed by X and Y is called the *bipartite adjacency matrix* of G and is denoted by $B(G)$. To establish our main result, we need the following theorem. It is straightforward to see that it generalizes Theorem 6. We recall again that the rank of a bipartite graph is always even.

Theorem 7. *Let G be a reduced bipartite graph of rank $r \geq 6$ and order $n > c(r) = 3 \cdot 2^{r/2-2} + r/2$ with bipartition $\{X, Y\}$. Then $\min\{|X|, |Y|\} = r/2$.*

Proof. For simplicity, we set $\rho = r/2$. We proceed by induction on ρ . The assertion holds for $\rho = 3$ by Theorem 6. So assume that $\rho \geq 4$. It is clear that $\text{rank}(G) \leq 2 \min\{|X|, |Y|\}$ and hence $\min\{|X|, |Y|\} \geq \rho$. Towards a contradiction, suppose that $\min\{|X|, |Y|\} \geq \rho + 1$.

Let H be an induced subgraph of G with the maximum possible order such that $\text{rank}(H) < \text{rank}(G)$ and let $t = n - |V(H)|$. By Lemma 2 and since H is bipartite, $\text{rank}(H) = r - 2$. In view of Lemma 2 (iii), suppose that $\{v_1, v'_1\}, \dots, \{v_s, v'_s\}$ are the duplication classes of H , for some $s \geq 0$, where the labeling of vertices comes from Lemma 2 (iv). For simplicity, set $S = \{v_1, \dots, v_s\}$ and $S' = \{v'_1, \dots, v'_s\}$. We denote the number of isolated vertices of H by ϵ . Lemma 2 (iii) implies that $\epsilon \in \{0, 1\}$. Let K be the resulting graph after deleting the possible isolated vertex from $H - S'$ and put $k = |V(K)|$. Clearly, $\text{rank}(K) = \text{rank}(H) = r - 2$ and since K is reduced, $k \leq b(r - 2)$ by Theorem 6. Moreover, since $\alpha(G) \geq n/2$, Lemma 2 (i) and Lemma 3 imply that $t < (n + 3)/4$. It then follows from $n = k + s + t + \epsilon \geq c(r) + 1$ and $k \leq b(r - 2)$ that $s > 2^{\rho-4} - \rho/4 + 1$. This means that $s \geq 2$. Further, let T_1 and T_2 be the sets given in Lemma 2 (iv). We may assume that $V(G) \setminus V(H) \subseteq X$ and $S \cup S' \subseteq Y$. For this, assume with no loss of generality that $T_1 \cap X \neq \emptyset$ and let $x \in T_1 \cap X$. By Lemma 2 (iv), $x \in N(v_i)$, for $i = 1, \dots, s$, meaning that $S \subseteq Y$. Since any v_i has some neighbor in $X \setminus T$ and $\{v_i, v'_i\}$ is a duplication class in H , we conclude that $S' \subseteq Y$ and thus $V(G) \setminus V(H) \subseteq X$. Let $P = Y \cap V(K - S)$, $Q = X \cap V(K)$ and set $p = |P|$, $q = |Q|$. In Figure 1, we depict the structure of G when $\epsilon = 0$.

Since $N(v_1) \triangle N(v_2) \subseteq Q$, Lemma 2 (i) yields that

$$t \leq q. \quad (3)$$

If $t \geq 3$, then we may assume with no loss of generality that $|T_1| \geq 2$. By Lemma 2 (iv), $N(x) \triangle N(y) \subseteq P$, for two distinct vertices $x, y \in T_1$ and so by Lemma 2 (i), $t \leq p$. So, in general, we have $t \leq p + 2$. From $n \geq c(r) + 1$ and $k \leq b(r - 2)$, it follows that $s + t = n - k - \epsilon \geq 2^{\rho-2} + 3 - \epsilon$. Since the symmetric difference of neighborhoods of any two vertices in S is contained in Q and has size at least t by Lemma 2 (i), so Theorem 4 yields that

$$s \leq 2^{q-t+1} \quad (4)$$

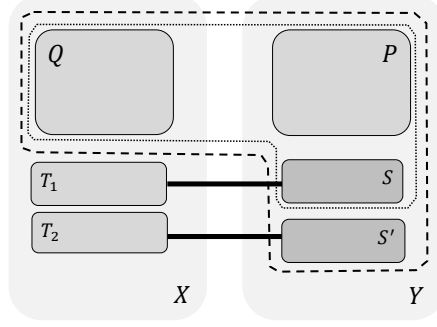


Figure 1: The structure of G concluded from Lemma 2 (The subgraphs $K \subset H$ are shown with dotted borders.)

and thus

$$2^{\rho-2} + 3 - \epsilon \leq s + t \leq 2^{q-t+1} + t. \quad (5)$$

We claim that $t = 2$ and $q = \rho - 1$. To establish the claim, we consider the following two cases.

Case 1. $k \leq (n + \rho - 3)/2$.

From $n = k + s + t + \epsilon$ and $k = p + q + s$, we have $p + q \leq t + \rho + \epsilon - 3$. If $t \geq 3$, then as we just showed, $t \leq p$ and thus in view of (3), we have $t \leq q \leq \rho + \epsilon - 3$. From (5), we find that $2^{\rho-2} + 2 \leq 2^{\rho-4} + \rho - 2$, which is impossible. Therefore $t \leq 2$. From $p + q \leq t + \rho + \epsilon - 3$ and $q + t = |X| \geq \rho + 1$, we obtain that $\rho + 1 - t \leq q \leq \rho + t - 2$ which in turn implies that $t = 2$ and either $q = \rho - 1$ or $q = \rho$. To get a contradiction, assume that $q = \rho$. Then $p + q \leq t + \rho + \epsilon - 3$ yields that $\epsilon = 1$ and $p = 0$. Since $P = \emptyset$, if one of T_1 or T_2 is empty, then the other one will be a duplication class of G by Lemma 2 (iv). Therefore both T_1 and T_2 are nonempty, since G is reduced. Hence we see that

$$B(G) = \begin{bmatrix} B(K) & B(K) & \mathbf{0} \\ \mathbf{j} & \mathbf{0} & 1 \\ \mathbf{0} & \mathbf{j} & 1 \end{bmatrix}.$$

Since $\text{rank}(B(K)) = \text{rank}(K)/2 = \rho - 1$, one can easily check that the rank of the row space of $B(G)$ is $\rho + 1$ which implies that $\text{rank}(G) = r + 2$, a contradiction. Therefore we must have $q = \rho - 1$, as claimed.

Case 2. $k > (n + \rho - 3)/2$.

Since $n \geq c(r) + 1$, we have $k > c(r - 2)$. By the induction hypothesis, $\min\{p + s, q\} = \rho - 1$. If $p + s = \rho - 1$, then from $p + 2 \geq t$, we find that

$$\rho - 1 = p + s = n - k + p - t - \epsilon \geq c(r) + 1 - b(r - 2) - 2 - \epsilon \geq 2^{\rho-2}$$

which is a contradiction to $\rho \geq 4$. Hence $q = \rho - 1$. Since $q + t = |X| \geq \rho + 1$, we deduce that $t \geq 2$. By (3), $t \leq \rho - 1$ and using (5), a straightforward calculation shows that $t = 2$, as claimed.

As we proved that $t = 2$ and $q = \rho - 1$, it follows from (5) that $\epsilon = 1$, implying that the equality occurs in (4). This means that the equality occurs in Theorem 4 for $n = \rho - 1$ and $d = 2$. Since K has no isolated vertex and $\rho \geq 4$, the cases (ii) and (iv) do not occur and so the induced subgraph on $Q \cup S$ is isomorphic to $\mathcal{O}_{\rho-1}$. If both T_1 and T_2 are nonempty, then $B(G)$ is of the form

$$\begin{bmatrix} B(\mathcal{O}_{\rho-1}) & B(\mathcal{O}_{\rho-1}) & \mathbf{0} & \star \\ \mathbf{j} & \mathbf{0} & 1 & \star \\ \mathbf{0} & \mathbf{j} & 1 & \star \end{bmatrix}.$$

Since $\text{rank}(B(\mathcal{O}_{\rho-1})) = \rho - 1$, we find that $\text{rank}(G) \geq r + 2$, a contradiction. So we may assume that T_2 is empty. Since the induced subgraph on $Q \cup S$ is isomorphic to $\mathcal{O}_{\rho-1}$, there exists a vertex $v \in S$ such that $|N(v) \cap Q| = 1$. If u is the isolated vertex of H , then $|N(u) \triangle N(v)| = 1$ which is impossible by Lemma 2 (i). This contradiction completes the proof. \square

4 Triangle-free graphs

In this section, we establish that every reduced non-bipartite triangle-free graph of rank r has at most $c(r)$ vertices. We also prove that there exists a unique reduced non-bipartite triangle-free graph of rank r and order $c(r)$.

Definition 8. For any integer $r \geq 4$, consider the graph $\mathcal{B}_{\lfloor r/2 \rfloor - 1}$ with bipartition $\{B, \mathcal{P}(B)\}$ and let $x \in B$. Let $N = N(x)$ and $M = \mathcal{P}(B) \setminus N$. For even r , we duplicate x and M to produce x' and M' . Now, introduce two new vertices y, z and join y to all vertices in $\{x, z\} \cup M$. For odd r , duplicate N and call it N' . Then introduce two new vertices y, z , join y to all vertices in $\{z\} \cup N$ and join z to all vertices in N' . We denote the resulting graph by \mathcal{C}_r . Clearly, the order of \mathcal{C}_r is $c(r)$. The graphs \mathcal{C}_8 and \mathcal{C}_9 are depicted in Figure 2.

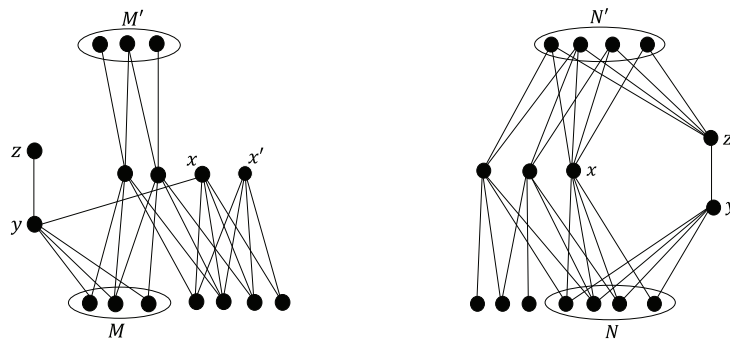


Figure 2: The graphs \mathcal{C}_8 (left) and \mathcal{C}_9 (right)

It is not hard to verify that one can define the graphs \mathcal{C}_r recursively as follows. Let \mathcal{C}_4 and \mathcal{C}_5 be the path and the cycle on 5 vertices, respectively. For $r = 6$ (respectively, $r = 7$), let A be a set of two vertices of distance 3 (respectively, 2) from each other in \mathcal{C}_4 (respectively, \mathcal{C}_5) and for $r \geq 8$, let A be

the maximum independent set of \mathcal{C}_{r-2} . Now, duplicate each vertex in A , introduce two new vertices u, v and join u to all vertices in $\{v\} \cup A$.

By the inductive definition of \mathcal{C}_r , it is easily seen that \mathcal{C}_r is a reduced non-bipartite triangle-free graph and $\text{rank}(\mathcal{C}_r) = \text{rank}(\mathcal{C}_{r-2}) + 2$. It follows that $\text{rank}(\mathcal{C}_r) = r$, for $r \geq 4$. Furthermore, we easily find from the definition of \mathcal{C}_r that

$$\alpha(\mathcal{C}_r) = 3 \cdot 2^{\lfloor r/2 \rfloor - 2} - 1, \quad (6)$$

for $r \geq 6$, and \mathcal{C}_r has a unique independent set of size $\alpha(\mathcal{C}_r)$.

Remark 9. Note that in Theorem 7, the hypothesis that $n > c(r)$ cannot be weakened. For any odd $r \geq 7$, if one removes the edge $\{y, z\}$ of \mathcal{C}_r , then the resulting graph, say H , is a reduced bipartite graph. Consider the graph $H - \{z\}$. Removing y from that results in the graph $\mathcal{B}_{(r-3)/2}$ with the neighborhood of x duplicated. So, $\text{rank}(H - \{y, z\}) = r - 3$ and clearly $\text{rank}(H - \{z\}) \leq r - 1$. Since $H - \{z\}$ is reduced, we must have from Lemma 1 (ii) that $\text{rank}(H - \{y, z\}) \leq \text{rank}(H - \{z\}) - 2$ and so $\text{rank}(H - \{z\}) = r - 1$. The sum of the row vectors corresponding to z and y in $A(H)$ is equal to that of x , so $\text{rank}(H) = \text{rank}(H - \{z\})$. Therefore, H is a reduced bipartite graph with bipartition $\{X, Y\}$ of rank $r - 1$ and order $c(r - 1)$ where $\min\{|X|, |Y|\} = (r + 1)/2$.

Theorem 10. *The order of a reduced non-bipartite triangle-free graph of rank r is at most $c(r) = 3 \cdot 2^{\lfloor r/2 \rfloor - 2} + \lfloor r/2 \rfloor$. Moreover, every reduced non-bipartite triangle-free graph of rank r and order $c(r)$ is isomorphic to \mathcal{C}_r .*

Proof. Let G be a reduced non-bipartite triangle-free graph of rank r and order $n \geq c(r)$. By induction on r , we prove that G is isomorphic to \mathcal{C}_r . In [1, 2], an algorithm is given to construct all reduced graphs of a given rank. We employed the algorithm and verified that the assertion holds for $r \leq 9$. The source code of our program can be found at <http://math.ipm.ac.ir/~tayfeh-r/Trianglefree.htm>. Hence let $r \geq 10$. For simplicity, we set $\rho = \lfloor r/2 \rfloor$. Let T be a subset of $V(G)$ with the minimum possible size such that $\text{rank}(G - T) \leq \text{rank}(G) - 2$. Put $H = G - T$ and $t = |T|$. We show that $t < (n + 3)/3$. If the minimum degree of G is less than $(n + 3)/3$, then we are done by Lemma 2 (i). Otherwise, since G is triangle-free, $\alpha(G) \geq (n + 3)/3$ and by Lemma 2 (i) and Lemma 3, we have

$$t \leq \frac{\frac{n+3}{3} \left(n - \frac{n+3}{3} \right)}{2 \left(\frac{n+3}{3} - 1 \right)} < \frac{n+3}{3},$$

as required. In view of Lemma 2 (iii), suppose that $\{v_1, v'_1\}, \dots, \{v_s, v'_s\}$ are the duplication classes of H , for some $s \geq 0$, where the labeling of vertices comes from Lemma 2 (iv). For simplicity, put $S = \{v_1, \dots, v_s\}$ and $S' = \{v'_1, \dots, v'_s\}$. Since G is triangle-free, by Lemma 2 (iv), $S \cup S'$ is an independent set. Denote the number of isolated vertices of H by ϵ . By Lemma 2 (iii), $\epsilon \in \{0, 1\}$. Let K be the resulting graph after deleting the possible isolated vertices from $H - S'$ and set $k = |V(K)|$. By Lemma 2, we have $\text{rank}(K) \geq r - 3$. Set $P = V(K) \setminus S$ and $p = |P|$. Further, let T_1 and T_2 be the sets given in Lemma 2 (iv) with sizes t_1 and t_2 , respectively. With no loss of generality, we assume that $t_1 \geq t_2$. We consider the following two cases.

Case 1. $k \leq c(r-2)$.

Let P_1 be the set of vertices in P which have a neighbor in S . Set $p_1 = |P_1|$ and $p_2 = |P \setminus P_1|$. For the structure of G when $\epsilon = 0$, see Figure 3. Since G is triangle-free, there is no edge between P_1 and

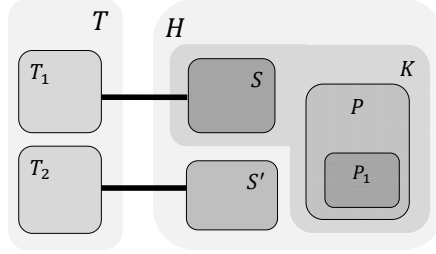


Figure 3: The structure of G in Case 1

T_1 . We have

$$s + t = n - k - \epsilon \geq 3 \cdot 2^{\rho-3} + 1 - \epsilon. \quad (7)$$

From $k = p + s \leq c(r-2)$ and (7), we see that

$$3 \cdot 2^{\rho-3} + 1 - \epsilon - t \leq s \leq 3 \cdot 2^{\rho-3} + \rho - 1 - p$$

which implies that

$$p \leq t + \rho + \epsilon - 2. \quad (8)$$

From $t < (n+3)/3$, $n \geq c(r)$ and $k \leq c(r-2)$, we find that $s = n - k - t - \epsilon > 2^{\rho-3} - \rho/3 - 1$ and so $s \geq 2$. Now, since $N(v_1) \triangle N(v_2) \subseteq P_1$,

$$t \leq p_1. \quad (9)$$

By (7), Lemma 2(i) and Theorem 4, we have

$$3 \cdot 2^{\rho-3} + 1 - \epsilon \leq s + t \leq 2^{p_1-t+1} + t. \quad (10)$$

Towards a contradiction, suppose that $t_1 \geq 2$. Then $t \leq |N(u) \triangle N(v)| \leq t_2 + p_2$, for each pair $u, v \in T_1$, and thus $t_1 \leq p_2$. If $\epsilon = 0$, then by (8) and (9), $t/2 \leq t_1 \leq p_2 \leq \rho - 2$ and hence $t \leq 2\rho - 4$. Moreover, it follows from (8) and $2 \leq t_1 \leq p_2$ that $p_1 \leq t + \rho - 4$. From (10), we conclude that $3 \cdot 2^{\rho-3} + 1 \leq 2^{\rho-3} + 2\rho - 4$, a contradiction. Therefore $\epsilon = 1$. By Lemma 2(ii), $N(u) \triangle N(v) \subseteq P \setminus P_1$, for any vertices $u, v \in T_1$, and hence $t \leq p_2$. Also, it follows from (8) and $t \leq p_2$ that $p_1 \leq \rho - 1$. Combining this with (9) gives $t \leq \rho - 1$, while combining with (10) gives $3 \cdot 2^{\rho-3} \leq 2^{\rho-t} + t$ which is a contradiction to $\rho \geq 5$. Thus $t_1 = 1$ and so $t_2 \leq 1$. Now we have

$$p_1 \geq \rho - 1, \quad (11)$$

since if $p_1 \leq \rho - 2$, then by (10), $3 \cdot 2^{\rho-3} \leq 2^{p_1} + 2 \leq 2^{\rho-2} + 2$ which is impossible for $\rho \geq 5$. We proceed to show that $t_2 = 0$. For this, we first establish the following property of K .

We show that if K is a bipartite graph with bipartition $\{K_1, K_2\}$, then S is contained in one of K_1 or K_2 . With no loss of generality, assume that $\ell = |P_1 \cap K_1| \leq p_1/2$. If $\ell = 0$, then $P_1 \subseteq K_2$, so that every vertex in S , being adjacent to a vertex in P_1 , must be in K_1 . Suppose $\ell \geq 1$. In order to get a contradiction, we first claim that $\ell = 1$. By Theorem 4, we obtain that

$$|K_1 \cap S| \leq 2^{p_1 - \ell - t + 1} \quad \text{and} \quad |K_2 \cap S| \leq 2^{\ell - t + 1}. \quad (12)$$

By (8), $p_1 \leq t - p_2 + \rho - 1$ and so $\ell \leq (\rho + 1)/2$. Using (7), (8) and (12), we find that

$$3 \cdot 2^{\rho-3} - t \leq s \leq 2^{p_1 - \ell - t + 1} + 2^{\ell - t + 1} \leq 2^{\rho - p_2 - \ell} + 2^\ell. \quad (13)$$

If $p_2 + \ell \geq 3$ and $\rho \geq 6$, then by (13),

$$3 \cdot 2^{\rho-3} - 2 \leq 2^{\rho - p_2 - \ell} + 2^\ell \leq 2^{\rho-3} + 2^{(\rho+1)/2} < 2^{\rho-3} + 2^{\rho-2} - 2,$$

a contradiction. If $p_2 + \ell = 2$ and $\rho \geq 6$, then $3 \cdot 2^{\rho-3} - 2 \leq 2^{\rho - p_2 - \ell} + 2^\ell \leq 2^{\rho-2} + 4$ which is again impossible. This implies that if $\rho \geq 6$, then $\ell = 1$. Now, assume that $\rho = 5$. By (13), we have

$$12 - t \leq s \leq 2^{p_1 - \ell - t + 1} + 2^{\ell - t + 1}. \quad (14)$$

Meanwhile, (8) gives

$$p_1 + p_2 \leq t + 4. \quad (15)$$

If $\ell \geq 3$, then by $\ell \leq p_1/2$, $t \leq 2$ and (15), we have $p_1 = 6$, $p_2 = 0$ and $t = 2$ which violate (14). Hence $\ell \leq 2$. If $\ell = 2$ and $t = 1$, then by (14) and (15), we see $p_1 = 5$, $p_2 = 0$ and the equality occurs in one of the inequalities of (12). By Theorem 4(i), K has an isolated vertex, a contradiction. Further, if $\ell = t = 2$, then by (14) and (15), we have $p_1 = 6$, $p_2 = 0$ and the equality occurs in both of the inequalities of (12). Since K is reduced, from Theorem 4(iii), one can deduce that the resulting graph after deleting all edges whose endpoints are in P_1 is isomorphic to the disjoint union of \mathcal{O}_2 and \mathcal{O}_4 . Since \mathcal{O}_2 is disjoint union of two edges and $\text{rank}(\mathcal{O}_4) = 8$, it is easily seen that $\text{rank}(K) \geq 12$ which contradicts $\text{rank}(K) \leq r - 2 \leq 9$. So we conclude that $\ell = 1$ and this completes the proof of the claim. Note that for any vertex $u \in K_2 \cap S$, we have $N(u) \cap V(K) \subseteq P_1 \cap K_1$. Since K is reduced and $\ell = |P_1 \cap K_1| = 1$, it follows that $K_2 \cap S$ has one element, say y . Letting $\{x\} = P_1 \cap K_1$, every duplication class of $K - \{x, y\}$ is contained in K_2 , since K is reduced and $N(y) \cap V(K) = \{x\}$. Also, every duplication class of $K - \{x, y\}$ has at most two elements, since otherwise, if u_1, u_2, u_3 belong to a duplication class, then at least two of them would be duplicates in K , a contradiction. If K' is the reduced graph corresponding to $K - \{x, y\}$, then by Lemma 1(i) we obtain that $\text{rank}(K') \leq \text{rank}(K) - 2 \leq r - 4$. Since $\text{rank}(K')$ is even, $\text{rank}(K') \leq 2\rho - 4$. By (11), we have $|P_1 \cap V(K')| \geq (\rho - 2)/2$ and therefore, using (7) and Theorem 6, we obtain that $(\rho - 2)/2 + 3 \cdot 2^{\rho-3} - 3 \leq |V(K')| \leq b(2\rho - 4)$ which is a contradiction to $\rho \geq 5$. This establishes the desired property of K .

Working towards a contradiction, suppose that $t_2 = 1$. By (10),

$$3 \cdot 2^{\rho-3} - 1 - \epsilon \leq s \leq 2^{p_1-1} \quad (16)$$

which yields that $\rho \leq p_1$. Meanwhile, by (8), we have $p_1 \leq \rho + \epsilon - p_2$. It follows that either $p_1 = \rho$ or $p_1 = \rho + 1$. First, assume that $p_1 = \rho$. The matrix $A(G)$ contains

$$\begin{array}{c} S \quad S' \quad P_1 \quad P_2 \quad T_1 \quad T_2 \\ \begin{array}{c} S \\ S' \\ P_1 \\ P_2 \\ T_1 \\ T_2 \end{array} \begin{bmatrix} \mathbf{0} & \mathbf{0} & B^\top & \mathbf{0} & \mathbf{j}^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & B^\top & \mathbf{0} & \mathbf{0} & \mathbf{j}^\top \\ B & B & \star & \star & \star & \star \\ \mathbf{0} & \mathbf{0} & \star & \star & \star & \star \\ \mathbf{j} & \mathbf{0} & \star & \star & \star & \star \\ \mathbf{0} & \mathbf{j} & \star & \star & \star & \star \end{bmatrix} \end{array} \quad (17)$$

as a principal submatrix. Since $\text{rank}(K) \geq r - 3$, the upper-left 4×4 block submatrix of (17) has rank at least $r - 3$. If \mathbf{j} is not contained in the row space of B , then the rank of (17) would be at least $r + 1$, a contradiction. Now, applying Lemma 5 to the column vectors of B , we find that $s \leq 5 \cdot 2^{\rho-4}$. If $\rho \geq 6$, this is less than $3 \cdot 2^{\rho-3} - 3$, contradicting (16). If $\rho = 5$, then $r \geq 10$, $s = 10$, $\epsilon = 1$. Hence, $2s + p_1 + p_2 + \epsilon + t = n \geq c(r) \geq c(10) = 29$ and $p_2 \leq \epsilon$. This gives $p_2 = 1$ and $k = s + p_1 + p_2 = 16 = c(8) = c(9)$. Thus K is isomorphic to either \mathcal{C}_8 or \mathcal{C}_9 . However, K contains the independent set S of size 10 in which $|N(u) \triangle N(v)| \geq 2$ for every distinct $u, v \in S$ while neither \mathcal{C}_8 nor \mathcal{C}_9 has such an independent set. Therefore $p_1 = \rho + 1$, $p_2 = 0$ and $\epsilon = 1$. Note that from (7) and $k = s + p_1 \leq c(r - 2)$, we have $s = 3 \cdot 2^{\rho-3} - 2$ and thus $k = c(r - 2)$. By the preceding paragraph, K is not bipartite, since otherwise G would be bipartite. Applying the induction hypothesis, K is isomorphic to either \mathcal{C}_{r-2} if $\text{rank}(K) = r - 2$ or \mathcal{C}_{r-3} if r is odd and $\text{rank}(K) = r - 3$. Hence, in view of (6), S is a maximal independent set of size $\alpha(K) - 1$ in K . To arrive at a contradiction, we show that \mathcal{C}_m has no maximal independent set of size $\alpha(\mathcal{C}_m) - 1$, for every integer $m \geq 8$. This can be directly checked when $m = 8$ or $m = 9$. For $m \geq 10$, we see that the degree of any vertex of \mathcal{C}_m not contained in the unique maximum independent set is at least $2^{\lfloor m/2 \rfloor - 2}$. Thus every independent set not contained in the unique maximum independent set is of size at most $c(m) - 2^{\lfloor m/2 \rfloor - 2} < \alpha(\mathcal{C}_m) - 1$. Therefore every independent set of size $\alpha(\mathcal{C}_m) - 1$ in \mathcal{C}_m is contained in the unique maximum independent set which means that \mathcal{C}_m has no maximal independent set of size $\alpha(\mathcal{C}_m) - 1$, as desired.

Therefore $t_2 = 0$. Again K is not bipartite, since otherwise G would be bipartite. It follows from (11) and $k = s + p_1 + p_2 \leq c(r - 2)$ that $s \leq 3 \cdot 2^{\rho-3} - p_2$. If $s = 3 \cdot 2^{\rho-3}$, then $p_2 = 0$, requiring that $k = c(r - 2)$. By the induction hypothesis, K is isomorphic to either \mathcal{C}_{r-2} or \mathcal{C}_{r-3} and so $\alpha(K) = 3 \cdot 2^{\rho-3} - 1$ which contradicts $s = 3 \cdot 2^{\rho-3}$. Hence (7) yields that $s = 3 \cdot 2^{\rho-3} - 1$ and $\epsilon = 1$. Then from $n \geq c(r)$, we have $p \geq \rho$ which in turn by (8) gives $p = \rho$ and so $k = s + p = c(r - 2)$. By the induction hypothesis, K is isomorphic to either \mathcal{C}_{r-2} or \mathcal{C}_{r-3} and so $\alpha(K) = 3 \cdot 2^{\rho-3} - 1$. This implies that $p_2 = 0$ and so $p_1 = \rho$. Now, the inductive definition of \mathcal{C}_r shows that G is isomorphic to \mathcal{C}_r .

Case 2. $k > c(r - 2)$.

By the induction hypothesis, K is a bipartite graph with bipartition, say $\{P_1 \cup S_1, P_2 \cup S_2\}$, where $P = P_1 \cup P_2$ and $S = S_1 \cup S_2$. Set $p_i = |P_i|$ and $s_i = |S_i|$, for $i = 1, 2$. With no loss of generality, we

may assume that $s_1 + p_1 \leq s_2 + p_2$. Since $\text{rank}(K) = 2\rho - 2$, Theorem 7 implies that

$$s_1 + p_1 = \rho - 1. \quad (18)$$

Let $S'_i = \{v'_j \mid v_j \in S_i, 1 \leq j \leq s\}$ for $i = 1, 2$. For the structure of G when $\epsilon = 0$, see Figure 4.

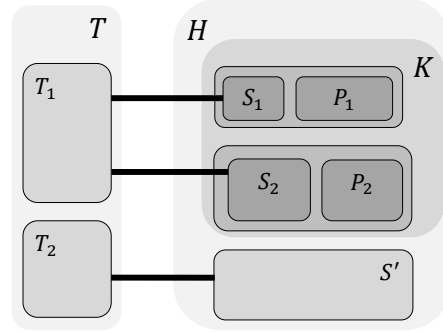


Figure 4: The structure of G in Case 2

Working towards a contradiction, suppose that $s_2 \leq 1$. We claim that $t \leq 2\rho - 2$. Assume that $s_1 \geq 1$. Since K is reduced, there exists a vertex $u \in P_2$ with a neighbor in S_1 . Since G is triangle-free, $N(u) \subseteq S_1 \cup S'_1 \cup P_1$ and so by Lemma 2 (i), we deduce that $t \leq 2s_1 + p_1 \leq 2\rho - 2$, as desired. Assume that $s_1 = 0$ and $s_2 = 1$. It is easily seen that the minimum degree among all vertices in $S_2 \cup S'_2$ does not exceed $t_2 + p_1$. By Lemma 2 (i), we find that $t \leq t/2 + \rho - 1$ and so $t \leq 2\rho - 2$, as required. Now, assume that $s_1 = s_2 = 0$. From $t < (n + 3)/3$, we find that

$$\begin{aligned} \alpha(G) &\geq p_2 + \epsilon \\ &= n - t - p_1 \\ &> n - \left(\frac{n}{3} + 1\right) - (\rho - 1) \\ &\geq 2^{\rho-1} - \frac{\rho}{3}. \end{aligned}$$

Therefore, $\alpha(G) \geq 15$. From $n - \alpha(G) \leq t + p_1 = t + \rho - 1$, Lemma 2 (i) and Lemma 3, we deduce that $t \leq \frac{15}{13}(\rho - 1)$. This establishes the claim. Now, by Theorem 6,

$$c(r) \leq n = k + t + s_1 + s_2 + \epsilon \leq b(2\rho - 2) + 3(\rho - 1) + 2$$

which implies that $\rho = 5$ and so $s_2 = n - k - s_1 - t - \epsilon \geq 10 - s_1 - t - \epsilon$. Hence

$$s_1 + t + \epsilon \geq 9. \quad (19)$$

First assume that $s_1 = 0$. Since $t \leq 2\rho - 2 = 8$, we conclude that $t = 8$, $\epsilon = 1$ and $s_2 = 1$. By Lemma 2, the vertices in $S_2 \cup S'_2$ have degree at least 8. On the other hand, the degree of any vertex of S_2 and S'_2 is at most $p_1 + t_1$ and $p_1 + t_2$, respectively. By (18), $p_1 = 4$ and as $t_1 + t_2 = 8$, we conclude that $t_1 = t_2 = 4$ and every vertex in P_1 is adjacent to every vertex in $S_2 \cup S'_2$. This shows that there is no edge between P_1 and T which in turn implies that G is bipartite, a contradiction. Now, suppose

that $s_1 \geq 1$. If $p_1 = 0$, then $s_2 = 0$ and so there is no edge between P_2 and T which again implies that G is bipartite, a contradiction. Hence $p_1 \geq 1$ and so by (18), $s_1 \leq 3$. This, in view of (19), implies that $t \geq 5$. There is a vertex $v \in P_2$ of degree 2 in K with a neighbor in S_1 . To see this, note that K is reduced and so we can view the vertices of $S_2 \cup P_2$ as distinct nonempty subsets of $S_1 \cup P_1$. If there does not exist such a vertex v , then $|S_2 \cup P_2| \leq 12$ implying that $k \leq 16$ which is impossible as $k > c(8) = 16$. Since v has a neighbor in S_1 and G is triangle-free, we deduce that $N(v) \subseteq S_1 \cup S'_1 \cup P_1$. It follows from Lemma 2 (i) that $t \leq 4$, contradicting (19). This contradiction establishes that $s_2 \geq 2$.

Since $n \geq c(r)$ and $k \leq b(2\rho - 2)$, we obtain that

$$s + t = n - k - \epsilon \geq 2^{\rho-2} + 2 - \epsilon. \quad (20)$$

For any pair $u, v \in S_2$, we have $t \leq |N(u) \triangle N(v)| \leq p_1$. By (20) and Theorem 4,

$$\begin{aligned} 2^{\rho-2} + 2 - \epsilon &\leq s + t \\ &= \rho - 1 - p_1 + s_2 + t \\ &\leq \rho - 1 + s_2 \\ &\leq \rho - 1 + 2^{p_1-t+1} \\ &= \rho - 1 + 2^{\rho-s_1-t}. \end{aligned}$$

Since $\rho \geq 5$, we have $s_1 + t \leq 2$. Towards a contradiction, assume that $t = 2$. Then $s_1 = 0$, so that $p_1 = \rho - 1$ by (18). If some $v \in P_1$ has a neighbor in T , then, since G is triangle-free, the neighborhood of each vertex in S_2 is a subset of $P_1 \setminus \{v\}$ and hence has size at most $p_1 - 1 = \rho - 2$. Thus by Theorem 4, $s_2 \leq 2^{\rho-3}$ which contradicts (20). So there is no edge between T and P_1 . Since G is not bipartite, there is an edge with endpoints in T . Since G is triangle-free, Lemma 2 (ii) implies that $\epsilon = 0$. From (20) and Theorem 4, we obtain that $s_2 = 2^{\rho-2}$. Since $n \geq c(r)$ and $k \leq b(2\rho - 2)$, we obtain that $p_2 = 2^{\rho-2} - 1$. By Theorem 4, the neighborhoods of vertices of P_2 (respectively, S_2) in P_1 correspond to odd-size (respectively, even-size) subsets of P_1 . Let $T = \{a_1, a_2\}$. Since G is triangle-free and there is an edge in T , we may assume that $T_1 = \{a_1\}$ and $T_2 = \{a_2\}$. If a_2 is adjacent to a vertex $x \in P_2$, then Theorem 4 (iii) implies that there exists a vertex $y \in S'_2$ such that $|N(x) \triangle N(y)| = 1$ which is impossible by Lemma 2 (i). Therefore $N(a_2) = S'_2$. Now $G - (S'_2 \cup \{a_2\})$ is a bipartite graph with bipartition $\{P_1 \cup \{a_1\}, S_2 \cup P_2\}$, and is reduced by Theorem 4. Since the number of vertices of $G - (S'_2 \cup \{a_2\})$ is larger than $b(2\rho - 2)$, Theorem 6 implies that $2\rho \leq \text{rank}(G - (S'_2 \cup \{a_2\}))$. On the other hand, by Lemma 1 (i), $\text{rank}(G - (S'_2 \cup \{a_2\})) \leq r - 2$. These give $2\rho \leq r - 2$ which is impossible.

Therefore $t = 1$. From $s_1 + t \leq 2$, we have $s_1 \leq 1$. Suppose that $s_1 = 0$. As G is not bipartite, there must be an edge between T and P_1 . So there is a vertex in P_1 with no neighbor in S_2 . Now, Theorem 4 and (20) imply that $s_2 = 2^{\rho-2}$. This is impossible since K is reduced. Hence $s_1 = 1$, so by (18), $p_1 = \rho - 2$ and as K is reduced, we clearly have $s_2 \leq 2^{\rho-2} - 1$. Also, by (20), we have $2^{\rho-2} - \epsilon \leq s_2$. Therefore, $s_2 = 2^{\rho-2} - 1$ and $\epsilon = 1$. Since $n \geq c(r)$ and $k \leq b(2\rho - 2)$, we have $p_2 = 2^{\rho-2}$, $n = c(r)$ and $k = b(2\rho - 2)$. Thus, by Theorem 6, K is isomorphic to $\mathcal{B}_{\rho-1}$. As $t = \epsilon = 1$, it is obvious that $\text{rank}(G) = \text{rank}(K) + 2$. Therefore r is even and the definition of \mathcal{C}_r shows that G is isomorphic to \mathcal{C}_r . \square

Acknowledgments

This research was in part supported by grants from IPM to the first author (No. 91050114) and the second author (No. 91050405). The authors thank anonymous referees for their valuable comments and suggestions which dramatically improved the presentation of the article.

References

- [1] S. AKBARI, P.J. CAMERON and G.B. KHOSROVSHAHI, Ranks and signatures of adjacency matrices, unpublished manuscript.
- [2] M.N. ELLINGHAM, Basic subgraphs and graph spectra, *Australas. J. Combin.* **8** (1993), 247–265.
- [3] E. GHORBANI, A. MOHAMMADIAN and B. TAYFEH-REZAIE, Maximum order of trees and bipartite graphs with a given rank, *Discrete Math.* **312** (2012), 3498–3501.
- [4] W.H. HAEMERS and M.J.P. PEETERS, The maximum order of adjacency matrices of graphs with a given rank, *Des. Codes Cryptogr.* **65** (2012), 223–232.
- [5] W.C. HUFFMAN and V. PLESS, *Fundamentals of Error-Correcting Codes*, Cambridge University Press, Cambridge, 2003.
- [6] A. KOTLOV, Rank and chromatic number of a graph, *J. Graph Theory* **26** (1997), 1–8.
- [7] A. KOTLOV and L. LOVÁSZ, The rank and size of graphs, *J. Graph Theory* **23** (1996), 185–189.
- [8] G.F. ROYLE, The rank of a cograph, *Electron. J. Combin.* **10** (2003), Note 11.